MTH 237-60

Extra credit 2. Rings and integral domains (1 pt)

As I said in the notes, a set of square matrices $M_n(\mathbf{R})$, like the integers, form a **ring** with the following rules:

1. Let A, B, and C be matrices. Addition is **associative**: $(A+B)+C = A+(B+C)$, meaning that it doesn't matter how you group your addends.

2. Addition is **commutative**: $A+B = B+A$, meaning that it doesn't matter how you order your addends.

3. There is an **additive identity**, a matrix of the appropriate size

$$0_{m\times n} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

   such that for any $A \in M_n(\mathbf{R})$, we have $A + 0_{m\times n} = 0_{m\times n} + A = A$.

4. Each $m \times n$ matrix A has an **additive inverse**, a matrix denoted $-A$ such that $A + (-A) = 0_{m\times n}$; in other words, subtraction is possible.

5. Multiplication is also **associative**: $(AB)C = A(BC)$.

6. If A is a $m \times n$ matrix, there exist square **multiplicative identities**

$$I_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

   and I_n such that $I_m A = A$ and $AI_n = A$. If A is square and $m = n$ is known from context, we just write I.

7. Multiplication **distributes** over addition:

$$(A+B)C = AC + BC$$

   and

$$A(B+C) = AB + AC.$$

Where does the similarity end? You may have intuited that there are a *lot* more square matrices with real entries than there are integers. The idea that there are "prime matrices" such that any matrix can be written as a product of these prime matrices may rightly strike you as incorrect. Therefore, the similarity has to end somewhere. It happens to end here:

Integers form an **integral domain**: you know that if $mn = 0$ for any integers m and n, at least one of m and n must be zero. (If a structure is an integral domain, that means it is like the integers in this way.) It turns out that $M_n(\mathbf{R})$ is not an integral domain, and so (for reasons you may learn in an abstract algebra course) it cannot be generated by a few "prime matrices."

For extra credit 2, prove items (1)-(7) about square matrices and then give an example of matrices A and B such that

$$AB = 0_{m,n}$$

where neither A nor B is the zero matrix.